

# A note on geodesic connectedness of Gödel type spacetimes\*

R. BARTOLO<sup>‡</sup>, A.M. CANDELA<sup>†</sup>, J.L. FLORES<sup>‡</sup>

<sup>‡</sup>Dipartimento di Matematica, Politecnico di Bari,  
Via E. Orabona 4, 70125 Bari, Italy  
`r.bartolo@poliba.it`

<sup>†</sup>Dipartimento di Matematica, Università degli Studi di Bari “A. Moro”,  
Via E. Orabona 4, 70125 Bari, Italy  
`candela@dm.uniba.it`

<sup>‡</sup>Departamento de Álgebra, Geometría y Topología,  
Facultad de Ciencias, Universidad de Málaga,  
Campus Teatinos, 29071 Málaga, Spain  
`floresj@agt.cie.uma.es`

## Abstract

In this note we reduce the problem of geodesic connectedness in a wide class of Gödel type spacetimes to the search of critical points of a functional naturally involved in the study of geodesics in standard static spacetimes. Then, by using some known accurate results on the latter, we improve previous results on the former.

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## Corresponding author:

Rossella Bartolo  
Dipartimento di Matematica, Politecnico di Bari  
Via E. Orabona 4, 70125 Bari, Italy  
e-mail: `r.bartolo@poliba.it`; fax: +39 080 5963612

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# 1 Introduction

In the last years the study of geodesic connectedness in some classes of Lorentzian manifolds has been carried out systematically by using variational methods (see [9] and references therein). This is the case of static and wave type spacetimes, where the variational methods yield optimal results by reducing the strongly indefinite action functional to a subtler Riemannian one (cf. [2, 5]).

In [7] a similar approach is applied to Gödel type spacetimes (here, Definition 3.1), but without providing optimal results (here, Theorem 3.3). In fact, in this case the variational result does not cover the classical Gödel Universe, whose geodesic connectedness is proved instead by a direct integration of the corresponding geodesic equations (see [7, Section 4]).

The aim of this paper is to improve meaningfully the result in [7] by a more careful application of variational methods. Indeed, we can deal with a functional similar to that one defined for standard static spacetimes (Section 4). Then, by using the accurate estimates in [2], we provide a substantial weakening of the boundedness assumptions about the metric coefficients in [7] (see Theorems 3.4 and 3.5). Unfortunately, the fact that our theorems do not cover the classical Gödel Universe seems to indicate that a sharp result cannot be reached only by using variational methods.

The paper is organized as follows: in Section 2, for the reader's convenience, we recall some definitions about the variational setting in the problem of geodesic connectedness; in Sections 3 and 4 we introduce the notions, and related results, of Gödel type and static spacetime, respectively; finally, in Section 5 we prove our main theorems.

# 2 Variational setting

In order to state our main results, firstly we recall some notations useful for the variational setting.

Taking a connected, finite-dimensional semi-Riemannian manifold  $(\mathcal{M}, g)$ , let  $H^1(I, \mathcal{M})$  be the set of curves  $z : I \rightarrow \mathcal{M}$ ,  $I = [0, 1]$ , such that for any local chart  $(U, \varphi)$  of  $\mathcal{M}$ , with  $U \cap z(I) \neq \emptyset$ , the curve  $\varphi \circ z$  belongs to the Sobolev space  $H^1(z^{-1}(U), \mathbb{R}^n)$ ,  $n = \dim \mathcal{M}$ . Then,  $H^1(I, \mathcal{M})$  is equipped with a structure of infinite-dimensional manifold modelled on the Hilbert space  $H^1(I, \mathbb{R}^n)$ . For any  $z \in H^1(I, \mathcal{M})$  the tangent space of  $H^1(I, \mathcal{M})$  at  $z$  can be written as follows:

$$T_z H^1(I, \mathcal{M}) = \{\zeta \in H^1(I, T\mathcal{M}) : \zeta(s) \in T_{z(s)}\mathcal{M} \text{ for all } s \in I\},$$

where  $T\mathcal{M}$  is the tangent bundle of  $\mathcal{M}$ .

If  $\mathcal{M}$  splits globally in the product of two semi-Riemannian manifolds  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , i.e.  $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ , then

$$H^1(I, \mathcal{M}) \equiv H^1(I, \mathcal{M}_1) \times H^1(I, \mathcal{M}_2)$$

and  $T_z H^1(I, \mathcal{M}) \equiv T_{z_1} H^1(I, \mathcal{M}_1) \times T_{z_2} H^1(I, \mathcal{M}_2)$  for all  $z = (z_1, z_2) \in \mathcal{M}$ .

On the other hand, if  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$  is a complete Riemannian manifold, it can be smoothly and isometrically embedded in an Euclidean space  $\mathbb{R}^N$  (cf. [12]). Hence,  $H^1(I, \mathcal{M}_0)$  is a submanifold of the Hilbert space  $H^1(I, \mathbb{R}^N)$ . In this case, we denote by  $d(\cdot, \cdot)$  the distance induced on  $\mathcal{M}_0$  by its Riemannian metric  $\langle \cdot, \cdot \rangle_R$ , i.e.

$$d(x_p, x_q) := \inf \left\{ \int_a^b \sqrt{\langle \dot{x}, \dot{x} \rangle_R} \, ds : x \in A_{x_p, x_q} \right\},$$

where  $x \in A_{x_p, x_q}$  if  $x : [a, b] \rightarrow \mathcal{M}_0$  is any piecewise smooth curve in  $\mathcal{M}_0$  joining  $x_p, x_q \in \mathcal{M}_0$ .

Given  $z_p, z_q \in \mathcal{M}$ , let us consider

$$\Omega^1(z_p, z_q) = \{z \in H^1(I, \mathcal{M}) : z(0) = z_p, z(1) = z_q\},$$

which is a (complete if  $\mathcal{M}$  is complete) submanifold of  $H^1(I, \mathcal{M})$ , having tangent space at any  $z \in \Omega^1(z_p, z_q)$  described as

$$T_z \Omega^1(z_p, z_q) = \{\zeta \in T_z H^1(I, \mathcal{M}) : \zeta(0) = 0 = \zeta(1)\}.$$

Moreover, for any  $l_p, l_q \in \mathbb{R}$ , let us denote

$$W(l_p, l_q) = \{l \in H^1(I, \mathbb{R}) : l(0) = l_p, l(1) = l_q\}.$$

Clearly,

$$W(l_p, l_q) = H_0^1(I, \mathbb{R}) + \bar{l},$$

with  $H_0^1(I, \mathbb{R}) = \{l \in H^1(I, \mathbb{R}) : l(0) = 0 = l(1)\}$ ,  $\bar{l} : s \in I \mapsto (1-s)l_p + sl_q \in \mathbb{R}$ . Hence,  $W(l_p, l_q)$  is a closed affine submanifold of the Hilbert space  $H^1(I, \mathbb{R})$  with tangent space

$$T_l W(l_p, l_q) = H_0^1(I, \mathbb{R}) \quad \text{for every } l \in W(l_p, l_q).$$

At last, let us recall a classical variational principle: if  $(\mathcal{M}, g)$  is a semi-Riemannian manifold, then  $\bar{z} : I \rightarrow \mathcal{M}$  is a geodesic joining two points  $z_p, z_q \in \mathcal{M}$  if and only if  $\bar{z} \in \Omega^1(z_p, z_q)$  is a critical point of the action functional

$$f(z) = \frac{1}{2} \int_0^1 g(z)[\dot{z}, \dot{z}] \, ds \quad \text{on } \Omega^1(z_p, z_q). \quad (2.1)$$

### 3 Gödel type spacetimes and statement of the main theorems

The classical *Gödel Universe* is an exact solution of the Einstein's field equations in which the matter takes the form of a rotating pressure-free perfect fluid. Mathematically, it is modelled by  $\mathbb{R}^4$  equipped with metric

$$ds^2 = dx_1^2 + dx_2^2 - \frac{1}{2} e^{2\sqrt{2}\omega x_1} dy^2 - 2 e^{\sqrt{2}\omega x_1} dy dt - dt^2, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where  $\omega > 0$  is the magnitude of the vorticity of the flow (see [11]). It can be proved that  $(\mathbb{R}^4, ds^2)$  is a geodesically connected Lorentzian manifold (cf. [7, Section 4]).

A natural generalization of the classical Gödel Universe is introduced in [7] as follows.

**Definition 3.1.** A Lorentzian manifold  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$  is a *Gödel type spacetime* if there exists a smooth (connected) finite-dimensional Riemannian manifold  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$  such that  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$  and the metric  $\langle \cdot, \cdot \rangle_L$  is described as:

$$\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_R + A(x)dy^2 + 2B(x)dydt - C(x)dt^2, \quad (3.1)$$

where  $x \in \mathcal{M}_0$ , the variables  $(y, t)$  are the natural coordinates of  $\mathbb{R}^2$  and  $A, B, C$  are  $C^1$  scalar fields on  $\mathcal{M}_0$  satisfying

$$\mathcal{H}(x) = B^2(x) + A(x)C(x) > 0 \quad \text{for all } x \in \mathcal{M}_0. \quad (3.2)$$

Note that condition (3.2) implies that metric (3.1) is Lorentzian. Furthermore, to this class it belongs not only the Gödel Universe, just taking

$$\begin{aligned} (\mathcal{M}_0, \langle \cdot, \cdot \rangle_R) &= (\mathbb{R}^2, dx_1^2 + dx_2^2), \\ A(x) &= -e^{2\sqrt{2}\omega x_1}/2, \quad B(x) = -e^{\sqrt{2}\omega x_1}, \quad C(x) \equiv 1, \end{aligned}$$

but also other physically relevant models of Lorentzian manifolds, as some stationary spacetimes, the Kerr–Schild spacetime or some warped product spacetimes (see [7] and references therein).

**Remark 3.2.** If the product  $A(x)C(x)$  is strictly positive on  $\mathcal{M}_0$ , the corresponding Gödel type spacetime reduces to a standard stationary one, and its geodesic connectedness has been deeply studied in previous works (see, for example, [1, 6]).

Every Gödel type spacetime  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$  admits two Killing vector fields,  $\partial_y$  and  $\partial_t$ , which are not necessarily timelike. At a first glance, the search of geodesics for these spacetimes can be handled in the same manner as in the static case: in fact, geodesics are critical points of the corresponding action functional which, as in the static case, becomes equivalent to a suitable simpler “Riemannian” one (cf. [7, Proposition 2.2]; here Section 5).

Taking a Gödel type spacetime  $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$  according to Definition 3.1, for each  $x \in H^1(I, \mathcal{M}_0)$  let us introduce the following notations:

$$\begin{aligned} a(x) &= \int_0^1 \frac{A(x)}{\mathcal{H}(x)} ds, \quad b(x) = \int_0^1 \frac{B(x)}{\mathcal{H}(x)} ds, \quad c(x) = \int_0^1 \frac{C(x)}{\mathcal{H}(x)} ds, \quad (3.3) \\ \mathcal{L}(x) &= b^2(x) + a(x)c(x). \end{aligned}$$

Then, the following result on geodesic connectedness in  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$  is obtained (see [7, Theorem 1.3], [8, Theorem 1.3]):

**Theorem 3.3.** *Let  $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$  be a Gödel type spacetime such that*

*(h<sub>1</sub>)  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$  is a complete Riemannian manifold;*

*(h<sub>2</sub>)  $|\mathcal{L}(x)| > 0$  for all  $x \in H^1(I, \mathcal{M}_0)$ ;*

*(h<sub>3</sub>) there exist  $k_1, k_2, k_3 > 0$  such that*

$$\left| \frac{a(x)}{\mathcal{L}(x)} \right| \leq k_1, \quad \left| \frac{b(x)}{\mathcal{L}(x)} \right| \leq k_2, \quad \left| \frac{c(x)}{\mathcal{L}(x)} \right| \leq k_3 \quad \text{for all } x \in H^1(I, \mathcal{M}_0).$$

*Then,  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$  is geodesically connected.*

Even if the hypotheses of this theorem are not optimal, the counterexample in [7, Appendix B] shows that they are reasonable. However, the boundedness assumptions in Theorem 3.3 can be improved considerably. As a matter of fact, preserving (h<sub>1</sub>), assumption (h<sub>3</sub>) can be replaced by heavily weaker hypotheses which directly involve the coefficients of the Lorentzian metric on  $\mathcal{M}$ . In fact, the main aim of this paper is to prove the following results:

**Theorem 3.4.** *Let  $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$  be a Gödel type spacetime such that (h<sub>1</sub>) holds. Moreover, assume that*

*(h'<sub>2</sub>) there exists  $\nu > 0$  such that  $\mathcal{L}(x) \geq \nu > 0$  for all  $x \in H^1(I, \mathcal{M}_0)$ ;*

*(h'<sub>3</sub>)  $A(x) - C(x) > 0$  for all  $x \in \mathcal{M}_0$ , and there exist  $\lambda \geq 0, k \in \mathbb{R}$  and a point  $\bar{x} \in \mathcal{M}_0$  such that the (positive) map*

$$\gamma : x \in \mathcal{M}_0 \mapsto \frac{\mathcal{H}(x)}{A(x) - C(x)} \in \mathbb{R}$$

*satisfies*

$$\gamma(x) \leq \lambda d^2(x, \bar{x}) + k \quad \text{for all } x \in \mathcal{M}_0. \quad (3.4)$$

*Then,  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$  is geodesically connected.*

**Theorem 3.5.** *Let  $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$  be a Gödel type spacetime such that (h<sub>1</sub>) holds. Moreover, assume that*

*(h''<sub>2</sub>) there exists  $\nu > 0$  such that  $\mathcal{L}(x) \leq -\nu < 0$  for all  $x \in H^1(I, \mathcal{M}_0)$ ;*

*(h''<sub>3</sub>)  $A(x) - C(x) < 0$  for all  $x \in \mathcal{M}_0$ .*

*Then,  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$  is geodesically connected.*

## 4 Static spacetimes

The proof of Theorem 3.4 relies on some results of variational nature coming from the study of geodesics in *static spacetimes*, i.e. Lorentzian manifolds endowed with an irrotational timelike Killing vector field. This section is dedicated to recall these statements.

**Definition 4.1.** A Lorentzian manifold  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$  is a *standard static spacetime* if there exists a smooth (connected) finite-dimensional Riemannian manifold  $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$  such that  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$  and the metric  $\langle \cdot, \cdot \rangle_L$  is described as:

$$\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_R - \beta(x) dt^2, \quad (4.1)$$

with  $x \in \mathcal{M}_0$ ,  $t$  the natural coordinate of  $\mathbb{R}$  and  $\beta$  a smooth strictly positive scalar field on  $\mathcal{M}_0$ .

The following two statements about geodesic connectedness in static spacetimes are well-known:

- The problem of geodesic connectedness in a (connected) static spacetime can be reduced to the same problem in a suitable standard static spacetime (see [2, Section 2]);
- Two points  $z_p = (x_p, t_p)$ ,  $z_q = (x_q, t_q)$  of a standard static spacetime  $(\mathcal{M}_0 \times \mathbb{R}, \langle \cdot, \cdot \rangle_L)$  are connected by a geodesic  $z = (x, t)$ , which is a critical point of the strongly indefinite action functional  $f$  in (2.1), with  $g = \langle \cdot, \cdot \rangle_L$  as in (4.1) and  $\Omega^1(z_p, z_q) = \Omega^1(x_p, x_q) \times W(t_p, t_q)$ , if and only if the functional

$$J(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle_R ds - \frac{\Delta_t^2}{2} \left( \int_0^1 \frac{1}{\beta(x)} ds \right)^{-1}, \quad (4.2)$$

with  $\Delta_t := t_p - t_q$ , admits a critical point on  $\Omega^1(x_p, x_q)$  (see [4]). This variational principle is a consequence of the existence of the Killing vector field  $\partial_t$  on  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ , which implies the constancy of  $\langle \partial_t, \dot{z} \rangle_L$  along each geodesic  $z$  on  $\mathcal{M}$ .

The existence of critical points for functional  $J$  in  $\Omega^1(x_p, x_q)$ , and thus the geodesic connectedness of standard static spacetimes, is ensured under different conditions for the growth of the metric coefficient  $\beta$ : when  $\beta$  is bounded (cf. [4]), when it is subquadratic (e.g., cf. [10]), and when it grows at most quadratically with respect to the distance  $d(\cdot, \cdot)$  induced on  $\mathcal{M}_0$  by its Riemannian metric  $\langle \cdot, \cdot \rangle_R$ . More precisely ([2, Theorem 1.1]):

**Theorem 4.2.** *Let  $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}, \langle \cdot, \cdot \rangle_L)$  be a standard static spacetime such that*

*$(H_1)$   $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$  is a complete Riemannian manifold,*

(H<sub>2</sub>) the positive function  $\beta$  grows at most quadratically at infinity, i.e. there exist  $\lambda \geq 0$ ,  $k \in \mathbb{R}$  and a point  $\bar{x} \in \mathcal{M}_0$  such that

$$\beta(x) \leq \lambda d^2(x, \bar{x}) + k \quad \text{for all } x \in \mathcal{M}_0.$$

Then,  $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$  is geodesically connected.

In this theorem the growth assumption for  $\beta$  is optimal. More precisely, there exists a family of geodesically disconnected static spacetimes with superquadratic, but arbitrarily close to quadratic, coefficients  $\beta$  ([2, Section 7]).

In order to prove Theorem 4.2, the following classical critical point result is applied (e.g., cf. [13, Theorem 2.7]):

**Theorem 4.3.** *Assume that  $\Omega$  is a complete Riemannian manifold and  $\mathcal{F}$  is a  $C^1$  functional on  $\Omega$  which satisfies the Palais–Smale condition, i.e. any sequence  $(x_k)_k \subset \Omega$  such that*

$$(\mathcal{F}(x_k))_k \text{ is bounded} \quad \text{and} \quad \lim_{k \rightarrow +\infty} \mathcal{F}'(x_k) = 0$$

*converges in  $\Omega$ , up to subsequences. Then, if  $\mathcal{F}$  is bounded from below, it attains its infimum.*

In fact, in our case, (H<sub>1</sub>) implies that  $\Omega = \Omega^1(x_p, x_q)$  is complete for each  $x_p, x_q \in \mathcal{M}_0$ . Moreover, the boundedness and the Palais–Smale conditions for  $\mathcal{F} = J$  are ensured by the following technical result (cf. [2, Propositions 4.1, 4.3]):

**Proposition 4.4.** *Under the hypotheses of Theorem 4.2, for each  $x_p, x_q \in \mathcal{M}_0$  the functional  $J$  on  $\Omega^1(x_p, x_q)$  is*

- *bounded from below;*
- *coercive, i.e.  $J(x) \rightarrow +\infty$  as  $\|\dot{x}\|^2 := \int_0^1 \langle \dot{x}, \dot{x} \rangle_R \, ds \rightarrow +\infty$ .*

## 5 On functional $\mathcal{J}$ and proofs of the main theorems

In [7] the authors develop a variational principle which allows one to study the geodesic connectedness of Gödel type spacetimes by finding critical points of a suitable functional  $\mathcal{J}$  (see (5.3) below). After recalling this principle, in the present section we rewrite functional  $\mathcal{J}$  to find a connection with the static functional  $J$  in (4.2). As a consequence, the geodesic connectedness of certain Gödel type spacetimes is deduced as a corollary of Theorem 4.2.

Throughout this section,  $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$  is a Gödel type spacetime according to Definition 3.1. Fixing  $z_p = (x_p, y_p, t_p)$ ,  $z_q = (x_q, y_q, t_q) \in \mathcal{M}$ , with  $x_p, x_q \in \mathcal{M}_0$  and  $(y_p, t_p), (y_q, t_q) \in \mathbb{R}^2$ , from the product structure of  $\mathcal{M}$  and the remarks in Section 2, we have that  $\bar{z} : I \rightarrow \mathcal{M}$  is a geodesic joining  $z_p$  to  $z_q$  in

$\mathcal{M}$  if and only if  $\bar{z}$  is a critical point of the  $C^1$  action functional in (2.1) with  $g = \langle \cdot, \cdot \rangle_L$  as in (3.1) and  $\Omega^1(z_p, z_q) = \Omega^1(x_p, x_q) \times W(y_p, y_q) \times W(t_p, t_q)$ .

One can take advantage of the Killing vector fields  $\partial_y, \partial_t$  on  $\mathcal{M}$  for proving a new variational principle free from the strongly indefinite character of  $f$  in  $\Omega^1(z_p, z_q)$ . In fact, for all  $s \in I$  and every  $x \in \Omega^1(x_p, x_q)$  such that  $\mathcal{L}(x) \neq 0$ , consider

$$\begin{aligned} \phi_y(x)(s) &:= y_p + \frac{\Delta_y b(x) - \Delta_t c(x)}{\mathcal{L}(x)} \int_0^s \frac{B(x)}{\mathcal{H}(x)} d\sigma \\ &\quad + \frac{\Delta_y a(x) + \Delta_t b(x)}{\mathcal{L}(x)} \int_0^s \frac{C(x)}{\mathcal{H}(x)} d\sigma, \end{aligned} \quad (5.1)$$

$$\begin{aligned} \phi_t(x)(s) &:= t_p - \frac{\Delta_y b(x) - \Delta_t c(x)}{\mathcal{L}(x)} \int_0^s \frac{A(x)}{\mathcal{H}(x)} d\sigma \\ &\quad + \frac{\Delta_y a(x) + \Delta_t b(x)}{\mathcal{L}(x)} \int_0^s \frac{B(x)}{\mathcal{H}(x)} d\sigma \end{aligned} \quad (5.2)$$

with

$$\Delta_y := y_q - y_p, \quad \Delta_t := t_q - t_p.$$

Standard arguments imply that

$$\phi_y : \Omega^1(x_p, x_q) \rightarrow W(y_p, y_q) \quad \text{and} \quad \phi_t : \Omega^1(x_p, x_q) \rightarrow W(t_p, t_q)$$

are  $C^1$  functions.

Then, one can establish the following proposition (see [7, Proposition 2.2] for further details):

**Proposition 5.1.** *If  $x_p, x_q \in \mathcal{M}_0$  are such that  $|\mathcal{L}(x)| > 0$  for all  $x \in \Omega^1(x_p, x_q)$ , then the following statements are equivalent:*

- (i)  $\bar{z} \in Z$  is a critical point of the action functional  $f$  in (2.1);
- (ii) setting  $\bar{z} = (\bar{x}, \bar{y}, \bar{t})$ , we have that  $\bar{x} \in \Omega^1(x_p, x_q)$  is a critical point of the  $C^1$  functional

$$\mathcal{J}(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle_R ds + \frac{\Delta_y^2 a(x) + 2\Delta_y \Delta_t b(x) - \Delta_t^2 c(x)}{2\mathcal{L}(x)} \quad (5.3)$$

on  $\Omega^1(x_p, x_q)$ , and the other components satisfy  $\bar{y} = \phi_y(\bar{x})$ ,  $\bar{t} = \phi_t(\bar{x})$ , with  $\phi_y, \phi_t$  as above.

Furthermore,

$$\mathcal{J}(x) = f(x, \phi_y(x), \phi_t(x)) \quad \text{for all } x \in \Omega^1(x_p, x_q). \quad (5.4)$$

Now, we are ready to develop the key point of our approach by writing functional (5.3) in a smarter way.



Giving  $x \in \Omega^1(x_p, x_q)$  such that  $|\mathcal{L}(x)| > 0$ , the numerator of the fraction in (5.3) is a quadratic form that can be rewritten as follows:

$$\Delta_y^2 a(x) + 2\Delta_y \Delta_t b(x) - \Delta_t^2 c(x) = \begin{pmatrix} \Delta_y & \Delta_t \end{pmatrix} \begin{pmatrix} a(x) & b(x) \\ b(x) & -c(x) \end{pmatrix} \begin{pmatrix} \Delta_y \\ \Delta_t \end{pmatrix}.$$

Note that the symmetric matrix

$$S(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & -c(x) \end{pmatrix}, \quad \text{with} \quad \det S(x) = -\mathcal{L}(x) \neq 0,$$

admits two real (non-null) eigenvalues

$$\begin{aligned} \lambda_{\pm}(x) &= \frac{a(x) - c(x) \pm \sqrt{(a(x) - c(x))^2 + 4\mathcal{L}(x)}}{2} \\ &= \frac{a(x) - c(x) \pm \sqrt{(a(x) + c(x))^2 + 4b^2(x)}}{2} \end{aligned} \quad (5.5)$$

which are the solutions of the characteristic equation

$$\lambda^2 - (a(x) - c(x))\lambda - \mathcal{L}(x) = 0. \quad (5.6)$$

Moreover, the following relations hold:

$$\lambda_-(x) \leq \lambda_+(x), \quad \lambda_-(x)\lambda_+(x) = -\mathcal{L}(x) \neq 0, \quad \lambda_+(x) + \lambda_-(x) = a(x) - c(x).$$

These eigenvalues are associated to the normalized eigenvectors:

$$\tilde{v}_{\pm}(x) = \frac{v_{\pm}(x)}{|v_{\pm}(x)|} \quad \text{with} \quad v_{\pm}(x) = \left( \frac{\lambda_{\pm}(x) + c(x)}{b(x)}, 1 \right) \quad \text{if } b(x) \neq 0,$$

$\tilde{v}_+(x) = (1, 0)$ ,  $\tilde{v}_-(x) = (0, 1)$  if  $b(x) = 0$  and  $a(x) > -c(x)$  (being  $\lambda_+(x) = a(x)$ ,  $\lambda_-(x) = -c(x)$ ), or  $\tilde{v}_+(x) = (0, 1)$ ,  $\tilde{v}_-(x) = (1, 0)$  if  $b(x) = 0$  and  $a(x) < -c(x)$  (being  $\lambda_+(x) = -c(x)$ ,  $\lambda_-(x) = a(x)$ ).

As a consequence, if  $D(x)$  is the matrix whose columns are  $\tilde{v}_{\pm}(x)$ , then

$$D(x)^{-1} = D(x)^T \quad \text{and} \quad D(x)^T S(x) D(x) = \begin{pmatrix} \lambda_+(x) & 0 \\ 0 & \lambda_-(x) \end{pmatrix}.$$

At any case, it results

$$\begin{aligned} & \begin{pmatrix} \Delta_y & \Delta_t \end{pmatrix} S(x) \begin{pmatrix} \Delta_y \\ \Delta_t \end{pmatrix} \\ &= \begin{pmatrix} \Delta_y & \Delta_t \end{pmatrix} D(x) \begin{pmatrix} \lambda_+(x) & 0 \\ 0 & \lambda_-(x) \end{pmatrix} D(x)^T \begin{pmatrix} \Delta_y \\ \Delta_t \end{pmatrix} \\ &= \begin{pmatrix} \Delta_+(x) & \Delta_-(x) \end{pmatrix} \begin{pmatrix} \lambda_+(x) & 0 \\ 0 & \lambda_-(x) \end{pmatrix} \begin{pmatrix} \Delta_+(x) \\ \Delta_-(x) \end{pmatrix} \\ &= \lambda_+(x)\Delta_+^2(x) + \lambda_-(x)\Delta_-^2(x), \end{aligned}$$

where  $\Delta_+(x) := (\Delta_y \ \Delta_t) \cdot \tilde{v}_+(x)$  and  $\Delta_-(x) := (\Delta_y \ \Delta_t) \cdot \tilde{v}_-(x)$ . By definition, we have

$$|\Delta_+(x)| \leq \sqrt{\Delta_y^2 + \Delta_t^2}, \quad |\Delta_-(x)| \leq \sqrt{\Delta_y^2 + \Delta_t^2}. \quad (5.7)$$

Thus, we obtain

$$\begin{aligned} \mathcal{J}(x) &= \frac{1}{2} \|\dot{x}\|^2 + \frac{(\Delta_y \ \Delta_t) S(x) \begin{pmatrix} \Delta_y \\ \Delta_t \end{pmatrix}}{2\mathcal{L}(x)} \\ &= \frac{1}{2} \|\dot{x}\|^2 - \frac{\lambda_+(x)\Delta_+^2(x) + \lambda_-(x)\Delta_-^2(x)}{2\lambda_+(x)\lambda_-(x)} \\ &= \frac{1}{2} \|\dot{x}\|^2 - \frac{1}{2} \frac{\Delta_+^2(x)}{\lambda_-(x)} - \frac{1}{2} \frac{\Delta_-^2(x)}{\lambda_+(x)}. \end{aligned} \quad (5.8)$$

In order to discuss the boundedness and growth behavior of  $\mathcal{J}$  in (5.8), let us focus on equation (5.6). From Descartes' rule of sign, the following cases may occur:

	$\mathcal{L}(x)$	$a(x) - c(x)$			
(i)	$\mathcal{L}(x) > 0$	$a(x) - c(x) > 0$	$\implies$	$\lambda_-(x) < 0$	$\lambda_+(x) > 0$
(ii)	$\mathcal{L}(x) > 0$	$a(x) - c(x) < 0$	$\implies$	$\lambda_-(x) < 0$	$\lambda_+(x) > 0$
(iii)	$\mathcal{L}(x) < 0$	$a(x) - c(x) > 0$	$\implies$	$\lambda_-(x) > 0$	$\lambda_+(x) > 0$
(iv)	$\mathcal{L}(x) < 0$	$a(x) - c(x) < 0$	$\implies$	$\lambda_-(x) < 0$	$\lambda_+(x) < 0$

**Remark 5.2.** From (5.5), the equality  $\lambda_-(x) = \lambda_+(x)$  occurs when  $b(x) = 0$  and  $a(x) = -c(x)$ , and it implies  $\mathcal{L}(x) = -c(x)^2 < 0$ <sup>1</sup>. So, from previous table, condition  $\mathcal{L}(x) > 0$  implies  $\lambda_-(x) < 0 < \lambda_+(x)$ .

**Lemma 5.3.** Assume that hypothesis  $(h_1)$  holds. Fixing  $x_p, x_q \in \mathcal{M}_0$ , suppose that  $\mathcal{J}$  is coercive on  $\Omega^1(x_p, x_q)$  and  $\nu > 0$  exists such that

$$|\mathcal{L}(x)| \geq \nu \quad \text{for all } x \in \Omega^1(x_p, x_q). \quad (5.9)$$

Then,  $\mathcal{J}$  satisfies the Palais–Smale condition on  $\Omega^1(x_p, x_q)$ .

*Proof.* Let  $(x_k)_k \subset \Omega^1(x_p, x_q)$  be such that

$$(\mathcal{J}(x_k))_k \text{ is bounded} \quad \text{and} \quad \lim_{k \rightarrow +\infty} \mathcal{J}'(x_k) = 0. \quad (5.10)$$

As  $\mathcal{J}$  is coercive, (5.10) implies that  $(\|\dot{x}_k\|)_k$  is bounded; hence, there exists a compact subset  $K$  in  $\mathcal{M}_0$  such that  $x_k(s) \in K$  for all  $s \in I$  and all  $k \in \mathbb{N}$ .

---

<sup>1</sup>Note that this situation cannot occur when  $B(x) \equiv 0$  on  $\mathcal{M}_0$ . In fact, under this condition hypothesis (3.2) forces  $A(x), C(x)$  to have the same sign, and the same must happen for  $a(x), c(x)$ , in contradiction with the equality  $a(x) = -c(x)$ .

$\mathbb{N}$ . Therefore,  $(x_k)_k$  is bounded in  $H^1(I, \mathcal{M}_0)$ , thus in  $H^1(I, \mathbb{R}^N)$  (as  $\mathcal{M}_0$  is isometrically embedded in  $\mathbb{R}^N$ ), and there exists  $x \in H^1(I, \mathbb{R}^N)$  such that

$$x_k \rightharpoonup x \text{ weakly in } H^1(I, \mathbb{R}^N) \text{ and } x_k \rightarrow x \text{ uniformly in } I$$

(up to subsequences). Clearly, assumption  $(h_1)$  implies  $x \in \Omega^1(x_p, x_q)$ . Moreover, by [3, Lemma 2.1] there exist two sequences  $(\xi_k)_k$  and  $(\nu_k)_k$  in  $H^1(I, \mathbb{R}^N)$ , with  $\xi_k \in T_{x_k} \Omega^1(x_p, x_q)$ , such that

$$\begin{aligned} x_k - x &= \xi_k + \nu_k \text{ for all } k \in \mathbb{N}, \\ \xi_k &\rightharpoonup 0 \text{ weakly and } \nu_k \rightarrow 0 \text{ strongly in } H^1(I, \mathbb{R}^N). \end{aligned} \quad (5.11)$$

In order to prove that  $\xi_k \rightarrow 0$  strongly in  $H^1(I, \mathbb{R}^N)$ , consider  $y_k = \phi_y(x_k)$ ,  $t_k = \phi_t(x_k)$  and  $z_k = (x_k, y_k, t_k)$ . As the coefficients  $A, B, C$  in (3.1) are bounded in  $K$ , and  $\mathcal{H}$  in (3.2) is bounded far away from zero in  $K$ , then so are the sequences  $(a(x_k))_k$ ,  $(b(x_k))_k$  and  $(c(x_k))_k$ . Whence, from (5.1), (5.2) and (5.9), it follows that also  $(\dot{y}_k)_k$  and  $(\dot{t}_k)_k$  are bounded in  $L^2(I, \mathbb{R})$ . From (5.4) and (5.10) it follows

$$\mathcal{J}'(x_k)[\xi_k] = f'(z_k)[(\xi_k, 0, 0)] = o(1),$$

i.e.,

$$\begin{aligned} o(1) &= \int_0^1 \langle \dot{x}_k, \dot{\xi}_k \rangle \, ds + \frac{1}{2} \int_0^1 \langle \nabla A(x_k), \xi_k \rangle \, \dot{y}_k^2 \, ds \\ &\quad + \int_0^1 \langle \nabla B(x_k), \xi_k \rangle \, \dot{y}_k \dot{t}_k \, ds - \frac{1}{2} \int_0^1 \langle \nabla C(x_k), \xi_k \rangle \, \dot{t}_k^2 \, ds. \end{aligned}$$

So, (5.11) and previous remarks give

$$\begin{aligned} \int_0^1 \langle \nabla A(x_k), \xi_k \rangle \, \dot{y}_k^2 \, ds &= o(1), \quad \int_0^1 \langle \nabla B(x_k), \xi_k \rangle \, \dot{y}_k \dot{t}_k \, ds = o(1), \\ \int_0^1 \langle \nabla C(x_k), \xi_k \rangle \, \dot{t}_k^2 \, ds &= o(1), \quad \int_0^1 \langle \dot{x}, \dot{\xi}_k \rangle \, ds = o(1), \quad \int_0^1 \langle \dot{\nu}_k, \dot{\xi}_k \rangle \, ds = o(1). \end{aligned}$$

In conclusion, we obtain  $\int_0^1 \langle \dot{\xi}_k, \dot{\xi}_k \rangle \, ds = o(1)$ , which completes the proof.  $\square$

Now, we are ready to give the proofs of our main results.

*Proof of Theorem 3.4.* Fix  $z_p = (x_p, y_p, t_p)$ ,  $z_q = (x_q, y_q, t_q) \in \mathcal{M}$ , with  $x_p, x_q \in \mathcal{M}_0$  and  $(y_p, t_p), (y_q, t_q) \in \mathbb{R}^2$ . By hypotheses  $(h'_2)$  and  $(h'_3)$ , the case (i) in the table above holds for all  $x \in \Omega^1(x_p, x_q)$  (recall (3.2), (3.3)). Moreover,

$$\lambda_+(x) \geq \frac{a(x) - c(x)}{2} > 0. \quad (5.12)$$

Thus, by (5.8), (5.12), (3.3), the expression of  $\gamma$  in  $(h'_3)$  and (5.7), it is

$$\begin{aligned}
\mathcal{J}(x) &\geq \frac{1}{2} \|\dot{x}\|^2 - \frac{\Delta_-^2(x)}{a(x) - c(x)} \\
&= \frac{1}{2} \|\dot{x}\|^2 - \Delta_-^2(x) \left( \int_0^1 \frac{A(x) - C(x)}{\mathcal{H}(x)} ds \right)^{-1} \\
&= \frac{1}{2} \|\dot{x}\|^2 - \Delta_-^2(x) \left( \int_0^1 \frac{1}{\gamma(x)} ds \right)^{-1} \\
&\geq \frac{1}{2} \|\dot{x}\|^2 - \sqrt{\Delta_y^2 + \Delta_t^2} \left( \int_0^1 \frac{1}{\gamma(x)} ds \right)^{-1}.
\end{aligned}$$

Define

$$\bar{J}(x) := \frac{1}{2} \|\dot{x}\|^2 - \sqrt{\Delta_y^2 + \Delta_t^2} \left( \int_0^1 \frac{1}{\gamma(x)} ds \right)^{-1}.$$

Note that  $\bar{J}$  has the same form of the static functional  $J$  in (4.2). Moreover, from (3.2), the scalar field  $\gamma$  is strictly positive and satisfies hypothesis (3.4), which is analogous to condition  $(H_2)$  in Theorem 4.2. So, from Proposition 4.4, it follows that  $\mathcal{J}$  is bounded from below and coercive. Furthermore, Lemma 5.3 implies that  $\mathcal{J}$  satisfies the Palais–Smale condition on  $\Omega^1(x_p, x_q)$ . Thus, Theorem 4.3 applies, and a geodesic connecting  $z_p$  with  $z_q$  exists.  $\square$

*Proof of Theorem 3.5.* Fix  $z_p = (x_p, y_p, t_p)$ ,  $z_q = (x_q, y_q, t_q) \in \mathcal{M}$ , with  $x_p, x_q \in \mathcal{M}_0$  and  $(y_p, t_p), (y_q, t_q) \in \mathbb{R}^2$ . By hypotheses  $(h'_2)$  and  $(h'_3)$ , the case (iv) in the table above holds for all  $x \in \Omega^1(x_p, x_q)$ . Then, functional  $\mathcal{J}$  in (5.8) is not only bounded from below, but also coercive, as

$$\mathcal{J}(x) \geq \frac{1}{2} \|\dot{x}\|^2 \quad \text{for all } x \in \Omega^1(x_p, x_q).$$

Then, by hypothesis  $(h_1)$ , Lemma 5.3 applies, and  $\mathcal{J}$  satisfies the Palais–Smale condition on  $\Omega^1(x_p, x_q)$ . Finally, as  $\Omega^1(x_p, x_q)$  is also complete, Theorem 4.3 implies the existence of a critical point for  $\mathcal{J}$  on  $\Omega^1(x_p, x_q)$ . Hence, a geodesic connecting  $z_p$  with  $z_q$  exists.  $\square$

The same arguments which allow one to prove the global property of the geodesic connectedness as stated in Theorems 3.4 and 3.5, can be used for proving the existence of a geodesic joining two fixed points.

**Proposition 5.4.** *Let  $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$  be a Gödel type spacetime such that  $(h_1)$  holds and fix two points  $z_p = (x_p, y_p, t_p)$ ,  $z_q = (x_q, y_q, t_q) \in \mathcal{M}$  such that  $\mathcal{L}(x) \geq \nu > 0$  (resp.  $\mathcal{L}(x) \leq -\nu < 0$ ) for all  $x \in \Omega^1(x_p, x_q)$ . If  $(h'_3)$  (resp.  $(h''_3)$ ) holds, then  $z_p$  and  $z_q$  are geodesically connected.*

**Remark 5.5.** If case (ii) occurs for all  $x \in \Omega^1(x_p, x_q)$ , the opposite inequality for the difference  $a(x) - c(x)$  prevents to proceed as in the proof of Theorem 3.4 (cf. (5.12)).

On the other hand, if case (iii) occurs for all  $x \in \Omega^1(x_p, x_q)$ , then  $\lambda_+(x) \geq \lambda_-(x) > 0$ . Then, from (5.8)

$$\mathcal{J}(x) \geq \frac{1}{2} \|\dot{x}\|^2 - \frac{1}{2} \frac{(\Delta_+^2(x) + \Delta_-^2(x))}{\lambda_-(x)}.$$

Clearly, it is possible to give suitable conditions for  $\lambda_-(x)$  on  $\Omega^1(x_p, x_q)$  which ensure the coercivity of  $\mathcal{J}$ . Nevertheless, the expression of  $\lambda_-(x)$  makes hard the analytic formulation of these bounds.

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